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THE KLEIN BOTTLE AND MULTICOMMODITY FLOWS

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Let G be an eulerian graph embedded on the Klein bottle. Then the maximum number of pairwise edge-disjoint orientation-reversing circuits in G is equal to the minimum number of edges intersecting all orientation-reversing circuits. This generalizes a theorem of Lins for the projective plane. As consequences we derive results on disjoint paths in planar graphs, including theorems of Okamura and of Okamura and Seymour.

1. Introduction

In [5] we proved:

Theorem 1. *Let $G=(V,E)$ be a planar bipartite graph embedded in the plane. Let I_1 and I_2 be two of its faces. Then there exist pairwise edge-disjoint cuts $\delta(X_1), \ldots, \delta(X_t)$ so that for each two vertices v, w with $v, w \in bd(I_1)$ or $v, w \in bd(I_2)$, the distance in G from v to w is equal to the number of cuts $\delta(X_j)$ separating v and w.* ∎

For $X \subseteq V$, $\delta(X)$ denotes the set of edges with exactly one of its end points in X. Cut $\delta(X)$ is said to *separate* v and w if $v \neq w$ and $|\{v, w\} \cap X|=1$. We denote the boundary of I by $bd(I)$. Faces are considered as *open* regions.

In this paper we derive from Theorem 1 some new results on graphs embedded on the Klein bottle and on plane multicommodity flows, and some known results due to Okamura, Okamura and Seymour, and Lins.

2. Graphs on the Klein bottle

Let $G=(V,E)$ be a graph embedded on the Klein bottle. We can represent the Klein bottle as obtained from the 2-sphere by adding two cross-caps. A circuit C in G is called *orientation-preserving* if after one turn of C the meaning of 'left' and 'right' is unchanged. It is called *orientation-reversing* if after one turn of C the meaning of 'left' and 'right' is exchanged.

Thus a circuit is orientation-preserving if and only if it passes the cross-caps an even number of times. It is orientation-reversing if and only if it passes the cross-caps an odd number of times. Hence the orientation-reversing circuits form a "binary clut-

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ter" in the sense of Seymour [6]: if C_1, C_2, C_3 are (the edge sets of) orientation-reversing circuits, then the symmetric difference $C_1 \triangle C_2 \triangle C_3$ contains an orientation-reversing circuit.

This implies that the inclusion-wise minimal edge sets intersecting all orientation-reversing circuits are exactly the inclusion-wise minimal sets in

(1) $\{D \subseteq E \mid |D \triangle C| \text{ is odd for each orientation-reversing circuit } C\}$.

In fact, it follows from our results that the hypergraph of orientation-reversing circuits, as well as its blocker (1), have the weak MFMC-property (Seymour [6]).

3. The minimum length of an orientation-reversing circuit

We first derive from Theorem 1:

Theorem 2. *Let $G=(V, E)$ be a bipartite graph embedded on the Klein bottle. Then the minimum length of any orientation-reversing circuit in G is equal to the maximum number of pairwise disjoint edge sets each intersecting all orientation-reversing circuits.*

Proof. Clearly, the maximum is not larger than the minimum. To show equality, we may assume that each face of G is orientable, i.e., contains no cross-cap. Indeed, if a face contains a cross-cap, we can add to G a path over this cross-cap, in such a way that the graph remains bipartite and such that the minimum-length of an orientation-reversing circuit remains unchanged (by taking the path long enough).

Let C_1 be a minimum-length orientation-reversing circuit in G, say with length t_1. We 'cut open' the Klein bottle S along C_1. In this way we obtain a bordered surface S', with a 1-sphere C_1' as border, so that S arises from S' by identifying opposite points on C_1'. So S' is a Möbius strip. Let i: $S' \to S$ be the identification map. The graph $G' := i^{-1}[G]$ is a graph on S', where $C_1' = i^{-1}[C_1]$.

As each face of G is orientable, also each face of G' (in S') is orientable. Therefore, G' contains an orientation-reversing circuit (in S'). Let C_2 be a minimum-length orientation-reversing circuit in G', say with length t_2. We may assume that C_2 is edge-disjoint from C_1 (by adding parallel edges). Next we 'cut open' the Möbius strip S' along C_2. We now obtain a cylinder S'', with boundary two 1-spheres B_1 and B_2. (It is a deformed cylinder if B_1 and B_2 have points in common.) The Klein bottle S arises from S'' by identifying opposite points on B_1 and by identifying opposite points on B_2. Let i': $S'' \to S$ be the identification map, and let $G'' := (i')^{-1}[G]$. So G'' is a planar graph, embeddable in the plane $\mathbf{R}^2$, so that two of its faces I_1(= unbounded face) and I_2 have the following properties:

(2) (i) the boundary of I_1 is a circuit D_1 of length $2t_1$, and the boundary of I_2 is a circuit D_2 of length $2t_2$;
(ii) S arises from $\mathbf{R}^2 \setminus (I_1 \cup I_2)$ by identifying pairs of opposite points on D_1 and by identifying pairs of opposite points on D_2.

We may assume that $S'' = \mathbf{R}^2 \setminus (I_1 \cup I_2)$.

Since t_1 is the minimum length of an orientation-reversing circuit in G, each pair of opposite vertices on D_1 has distance exactly t_1. Since t_2 is the minimum length of an orientation-reversing circuit in G, each pair of opposite vertices on D_2 has distance exactly t_2.

By Theorem 1, there exist pairwise disjoint cuts $\delta(X_1), \ldots, \delta(X_t)$ so that for each two vertices v and w of G'' with $v, w \in bd(I_1)$ or $v, w \in bd(I_2)$, the distance in G'' from v to w is equal to the number of cuts $\delta(X_j)$ separating v and w. We may assume that each $\delta(X_j)$ separates at least one such pair v, w (all other cuts can be deleted), and that each $\delta(X_j)$ is a minimal nonempty cut (inclusion-wise).

Each cut $\delta(X_j)$ intersects any subpath P of D_1 of length t_1 at most once (as P is intersected by t_1 of the $\delta(X_j)$, as P is a shortest path between its two end points). So if $\delta(X_j)$ intersects D_1, it intersects D_1 exactly twice, in two opposite edges. Simitarly, if $\delta(X_j)$ intersects D_2, it intersects D_2 exactly twice, in two opposite edges.

We can classify the $\delta(X_j)$ into three classes:

(3) (i) those intersecting both D_1 and D_2, say $\delta(X_1), \ldots, \delta(X_s)$;
(ii) those intersecting D_1 but not D_2, say $\delta(X_{s+1}), \ldots, \delta(X_{t_1})$;
(iii) those intersecting D_2 but not D_1, say $\delta(X_{t_1+1}), \ldots, \delta(X_t)$.

Note that $t_2 = s + (t - t_1)$, and hence $s = t_1 + t_2 - t$.

First consider $\delta(X_1), \ldots, \delta(X_s)$. Each such $\delta(X_j)$ is (since it is a minimal cut) the set of edges of G'' intersected by two curves Γ_1 and Γ_2, where Γ_1 connects points p' on D_1 and p'' on D_2, while Γ_2 connects points q' on D_1 and q'' on D_2, in such a way that p' and q' are opposite on D_1, and p'' and q'' are opposite on D_2:

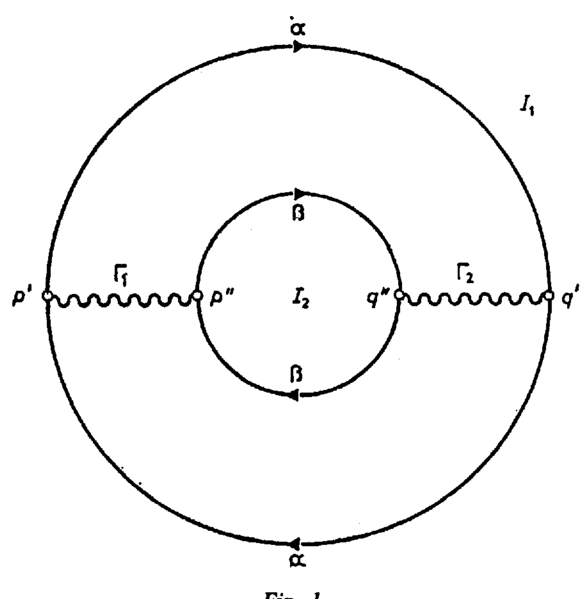

Fig. 1

The space $i'[S'' \setminus (\Gamma_1 \cup \Gamma_2)]$ is orientable, since it arises from Fig. 1 by identifying the two curves α (in the orientation given), and similarly the two curves β, which yields a cylinder. Hence $i'[\Gamma_1 \cup \Gamma_2]$ intersects all orientation-reversing closed curves on S, and hence $i'[\delta(X_j)]$ is a set of edges in G intersecting all orientation-reversing circuits.

Similarly, each set

(4) $$i'[\delta(X_{s+j})\cup\delta(X_{t_1+j})],$$

for $j=1, \ldots, t_1-s$, intersects all orientation-reversing circuits in G (note $t_1+(t_1-s)\leqq \leqq t_2+t_1-s=t$ as $t_1\leqq t_2$). Now $\delta(X_{s+j})$ is the set of edges intersected by a curve Γ_1 connecting two opposite points p' and q' on D_1, while $\delta(X_{t_1+j})$ is the set of edges intersected by a curve Γ_2 connecting two opposite points p'' and q'' on D_2:

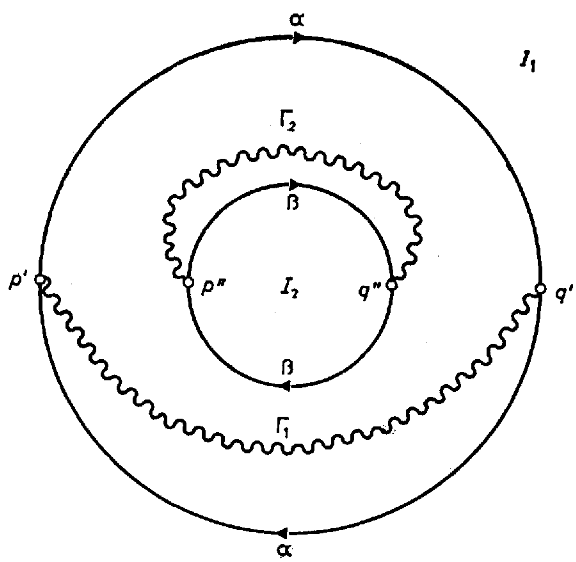

Fig. 2

Again the space $i'[S''\setminus(\Gamma_1\cup\Gamma_2)]$ is orientable, since it arises from Fig. 2 by identifying the two curves α and the two curves β, yielding again a cylinder. So $i'[\Gamma_1\cup\Gamma_2]$ intersects all orientation-reversing closed curves in S, and hence (4) intersects all orientation-reversing closed curves in S, and hence (4) intersects all orientation-reversing circuits in G.

Combining,

(5) $$i'[\delta(X_1)], \ldots, i'[\delta(X_s)], i'[\delta(X_{s+1})\cup\delta(X_{t_1+1})], \ldots, i'[\delta(X_{t_1})\cup\delta(X_{2t_1-s})]$$

are t_1 pairwise edge-disjoint sets of edges of G, each intersecting all orientation-reversing circuits. ▮

Note. In fact, the proof shows that it suffices to require that each nullhomotopic circuit in G is even (instead of G being bipartite). Indeed, this implies that the graph G'' described in the proof above is bipartite.

4. The max-flow min-cut property

Theorem 2 implies the following. Let $G=(V, E)$ be a graph embedded on the Klein bottle. Let

(6) $\mathscr{C}:=$ collection of orientation-reversing circuits in G;
$b(\mathscr{C}):=$ collection of edge-sets intersecting each orientation-reversing circuit in G.

Then the hypergraph $(E, b(\mathscr{C}))$ has the weak MFMC-property, in the sense of Seymour [6]. That is, the vertices of the polytope in $\mathbf{R}^E$ determined by:

(7)
$$\text{(i)}\quad 0 \leqq x(e) \leqq 1 \quad (e\in E),$$
$$\text{(ii)}\quad \sum_{e\in D} x(e) \geqq 1 \quad (D\in b(\mathscr{C})),$$

are $\{0, 1\}$-vectors. These vectors are exactly the characteristic vectors of subsets of E containing an orientation-reversing circuit.

This follows from the fact that, for any l: $E\to \mathbf{Z}_+\setminus\{0\}$, the minimum value of

(8)
$$\sum_{e\in E} l(e)x(e)$$

over (7) is achieved by an integer vector x. To see this, we may assume that $l(e)$ is even for each $e\in E$. Now replace each edge e of G by a path of length $l(e)$. We obtain a bipartite graph G'. Let C' be a minimum-length orientation-reversing circuit in G'. By Theorem 2 there exist pairwise disjoint edge sets $D_1', \ldots, D_t'$ in G' each intersecting all orientation-reversing circuits in G', so that t is equal to the number of edges in C'. Let $C, D_1, \ldots, D_t$ be the 'projections' of C', $D_1', \ldots, D_t'$ to G. Then

(9)
$$t = \sum_{e\in E} l(e)\chi^C(e),$$

where χ^C denotes the characteristic vector of C. Since $D_1, \ldots, D_t$ give a dual solution to (7) of value t, it follows that χ^C is an optimum solution.

By Lehman's theorem [1] the weak MFMC-property is maintained under taking blocking hypergraphs. So also $\mathscr{C}$ has the weak MFMC-property. That is, the vertices of the polytope in $\mathbf{R}^E$ determined by:

(10)
$$\text{(i)}\quad 0 \leqq x(e) \leqq 1 \quad (e\in E),$$
$$\text{(ii)}\quad \sum_{e\in C} x(e) \geqq 1 \quad (C\in\mathscr{C}),$$

are $\{0, 1\}$-vectors. These vectors are exactly the characteristic vectors of sets in $b(\mathscr{C})$. In the following section we show that a stronger property holds.

5. Packing orientation-reversing circuits

We derive from the previous results:

Theorem 3. *Let $G=(V, E)$ be an eulerian graph embedded on the Klein bottle. Then the maximum number of pairwise edge-disjoint orientation-reversing circuits is equal to the minimum number or edges intersecting all orientation-reversing circuits.*

Proof. Clearly, the maximum is not more than the minimum. Suppose equality does not hold, and let G form a counterexample with

$$\sum_{v\in V} 2^{\deg(v)} \tag{11}$$

as small as possible (where $\deg(v)$ denotes the degree of v). Let D be a set of edges intersecting all orientation-reversing circuits in G, of minimum size $t=|D|$. Since t is equal to the minimum value of

$$\sum_{e\in E} x(e) \tag{12}$$

over (10) (as (10) is the convex hull of the characteristic vectors of edge-sets intersecting all orientation-reversing circuits), there exist, by linear programming duality, orientation-reversing circuits $C_1, \ldots, C_k$ (pairwise different) and reals $\lambda_1, \ldots, \lambda_k > 0$, so that:

(13)

$$\text{(i)} \quad \sum_{i=1}^{k} \lambda_i = t,$$

$$\text{(ii)} \quad \sum_{i=1}^{k} \lambda_i \chi^{C_i}(e) \leqq 1 \quad (e\in E).$$

In fact, what we must show is that each λ_i can be taken to be 1.

Consider a vertex v of G, and the edges $e_1, \ldots, e_{2d}$ incident to v, in cyclic order:

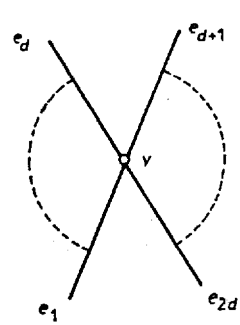

Fig. 3

Thus e_1 and e_{d+1} are 'opposite', and similarly e_2 and e_{d+2}, e_3 and e_{d+3}, ..., e_d and e_{2d}. We show that for each circuit C_i and each $j=1, \ldots, d$:

$$C_i \text{ passes } e_j \Leftrightarrow C_i \text{ passes } e_{d+j}. \tag{14}$$

Having shown this for each vertex v, each j and each C_i, it follows that the $C_1, \dots, C_k$ are pairwise edge-disjoint. Since $k \geqq t$ (as $\lambda_i \leqq 1$ for all i), this proves the theorem.

Suppose (14) does not hold for some v, i, j. Without loss of generality, $i=1$, $j=1$, and C_1 passes e_1 and e_m for some m with $2 \leqq m \leqq d$. Now replace Fig. 3. by:

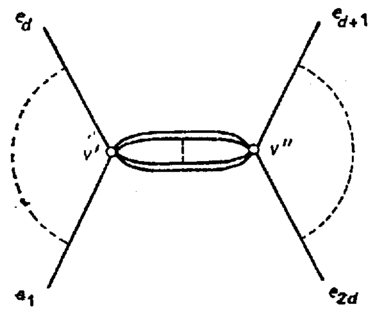

Fig. 4

where there are $d-2$ parallel edges connecting the new vertices v' and v''. Let G' be the new graph obtained. So G arises from G' by contracting the parallel edges connecting v' and v''. (If $d=2$, we identify v' and v''.) Graph G' is eulerian again, with sum (11) smaller than for G. So by the minimality hypothesis, the theorem to be proved holds for G'.

Let D' be a minimum-sized set of edges in G' intersecting all orientation-reversing circuits in G'. Let $t' := |D'|$. If $t' \geqq t$, G' would contain t pairwise edge-disjoint orientation-reversing circuits. After identifying v' and v'', this gives t pairwise edge-disjoint orientation-reversing circuits in G, contradicting our assumption. So $t' < t$.

We show $t' \leqq t-2$. Let $\bar{D}$ be the set of edges in G' corresponding to D. By the minimality of D, D intersects each orientation-reversing circuit in G an odd number of times, and each orientation-preserving circuit in G an even number of times. Hence also $\bar{D}$ intersects each orientation-reversing circuit in G' an odd number of times, and each orientation-preserving circuit in G' an even number of times. By the minimality of D', also D' has odd intersection with each orientation-reversing circuit, and even intersection with each orientation-preserving circuit in G'. This implies that the symmetric difference $\bar{D} \triangle D'$ has even intersection with each circuit in G'. So $\bar{D} \triangle D'$ is a cut in G', and hence, as G' is eulerian, $|\bar{D} \triangle D'|$ is even. That is, $|\bar{D}| \equiv |D'|$ (mod 2). Therefore, as $t' < t$, we know $t' \leqq t-2$.

Let π denote the set of parallel edges in G' connecting v' and v''. We show that $\pi \subseteq D'$. If not, $\pi \neq \emptyset$, and hence $d \geqq 3$. Let $e \in \pi \setminus D'$. Then $D' \setminus \pi$ intersects all orientation-reversing circuits in G', and hence (after contracting the edges in π) also all orientation-reversing circuits in G. However, $|D' \setminus \pi| \leqq |D'| < |D|$, contradicting the minimality of D.

Let

$$D'' := (D' \setminus \pi) \cup \{e_1, \dots, e_d\}. \tag{15}$$

Since $|\pi| = d-2$, we know $|D''| \leqq t'+2 \leqq t$. Let $\overline{D''}$ be the set of edges in G corresponding to D''. Then $\overline{D''}$ intersects all orientation-reversing circuits in G (since each orientation-reversing circuit in G intersects $\{e_1, ..., e_d\}$ or comes from an orientation-reversing circuit in G' not intersecting π). So $|\overline{D''}|=t$. Hence $\chi^{\overline{D''}}$ attains the minimum of (12) over (10). So by complementary slackness, $|\overline{D''}\cap C_1|=1$. This contradicts the fact that $e_1, e_m \in \overline{D''}\cap C_1$. ∎

Theorem 3 generalizes a theorem of Lins [2], which in fact is Theorem 3 with respect to the projective plane instead of the Klein bottle. If G is a graph embedded on the projective plane, we can insert a cross-cap in one of the faces of G. This transforms the projective plane to a Klein bottle. As the meaning of 'orientation-reversing' is not changed by this insertion (for the circuits in G), it reduces Lins' theorem to Theorem 3.

Theorem 3 cannot be extended to compact surfaces with more than two cross-caps, as we can embed K_5 on such a surface in such a way that the orientation-reversing circuits are exactly the odd-sized circuits. Then the maximum number of pairwise edge-disjoint orientation-reversing circuits is equal to 2, while not less than 4 edges are necessary to intersect all orientation-reversing circuits.

4. Plane multicommodity flows

From Theorem 3 we derive a new result on the existence of pairwise edge-disjoint paths in a planar graph. Let $G=(V, E)$ be a graph, and let $r_1, ..., r_k$, $s_1, ..., s_k$ be vertices of G. We consider the following two conditions:

(16) (parity condition): for each vertex v of G:

$$\deg(v)+|\{i\in\{1, ..., k\}|r_i = v\}|+|\{i\in\{1, ..., k\}|s_i = v\}| \text{ is even;}$$

(cut condition): for each $X\subseteq V$:

$$|\delta(X)|\geqq \text{number of pairs } r_i, s_i \text{ separated by } \delta(X).$$

Theorem 4. *Let $G=(V, E)$ be a planar graph embedded in the plane $\mathbf{R}^2$. Let $r_1, ..., r_k$, $s_1, ..., s_k$ be vertices of G satisfying the parity condition. Let $r_1, ..., r_k$ be incident to the unbounded face I_1 in clockwise order. Let $s_1, ..., s_k$ be incident to some other face I_2 in anti-clockwise order. Then there exist pairwise edge-disjoint paths $P_1, ..., P_k$ where P_i connects r_i and s_i $(i=1, ..., k)$, if and only if the cut condition is satisfied.*

Proof. Since the cut condition trivially is a necessary condition, we only show sufficiency. Let the cut condition be satisfied. We can extend $\mathbf{R}^2\setminus(I_1\cup I_2)$ to the Klein bottle, by adding a cylinder between the boundaries of I_1 and I_2. We can extend G to a graph G' on the Klein bottle adding edges $e_1, ..., e_k$ over this cylinder, so that e_i connects r_i and s_i $(i=1, ..., k)$. Then a circuit in G' is orientation-reversing if and only if it contains an odd number of edges from $e_1, ..., e_k$. So it suffices to show that G' contains k pairwise edge-disjoint orientation-reversing circuits.

By the parity condition, G' is eulerian. So we can apply Theorem 3. Hence it suffices to show that each set D of edges of G' intersecting all orientation-reversing circuit has size at least k. We may assume that D is a minimal set of edges in G' intersecting all orientation-reversing circuits in G'. Hence $|D\cap C|$ is even for each circuit C in G. Therefore, $D\cap E$ is a cut $\delta(X)$ in G. Now we have for each $i=1, ..., k$:

(17) $\delta(X)$ does not separate r_i and $s_i \Rightarrow e_i\in D$.

Indeed, if $\delta(X)$ does not separate r_i and s_i, then there exists a path P in G connecting r_i and s_i and containing an even number of edges in D. Now as $P\cup\{e_i\}$ is an orientation-reversing circuit, it intersects D an odd number of times, and hence $e_i\in D$.

Assertion (17) implies that $|D\cap\{e_1, \dots, e_k\}|$ is not less than the number of pairs r_i, s_i not separated by $\delta(X)$. Hence

$$(18)\qquad |D|=|D\cap E|+|D\cap\{e_1, \dots, e_k\}| \geqq |\delta(X)|+\text{number of pairs } r_i, s_i \text{ not separated by } \delta(X)\geqq k,$$

by the cut condition. ∎

5. A theorem of Okamura

One can also derive a theorem of Okamura [3]:

Theorem 5. *Let $G=(V, E)$ be a planar graph embedded in the plane $\mathbf{R}^2$. Let I_1 and I_2 be two of its faces, and let $r_1, \dots, r_k, s_1, \dots, s_k$ be vertices satisfying the parity condition, so that for each $i=1, \dots, k$: $r_i, s_i\in bd(I_1)$ or $r_i, s_i\in bd(I_2)$. Then there exist pairwise edge-disjoint paths $P_1, \dots, P_k$ where P_i connects r_i and s_i $(i=1, \dots, k)$, if and only if the cut condition is satisfied.*

Proof. Again, it suffices to show sufficiency. Without loss of generality, I_1 is the unbounded face, and $r_1, \dots, r_t, s_1, \dots, s_t\in bd(I_1)$ and $r_{t+1}, \dots, r_k$, $s_{t+1}, \dots, s_k\in bd(I_2)$. By an argument due to S. Lins, we may assume that $r_1, \dots, r_t, s_1, \dots, s_t$ occur in cyclic order around I_1. To see this, first note that we may assume that the vertices $r_1, \dots, r_k, s_1, \dots, s_k$ are distinct and have degree 1 (as we can add a new vertex of degree 1 to any r_i or s_i and replace this r_i or s_i by the new vertex). Call two pairs r_i, s_i and r_j, s_j on $bd(I_1)$ *crossing* if $i\neq j$ and r_i, r_j, s_i, s_j occur in this cyclic order around the boundary of I_1, clockwise or anti-clockwise. Suppose not all pairs of pairs r_i, s_i are crossing. Then there exist i, j so that r_i, s_i and r_j, s_j are non-crossing and so that there is no pair r_h, s_h on that part of the boundary of I_1 that connects r_i and r_j and that does not pass s_i and s_j. Now we can add in I_1 three new vertices w, r_i' and r_j' and four new edges as follows:

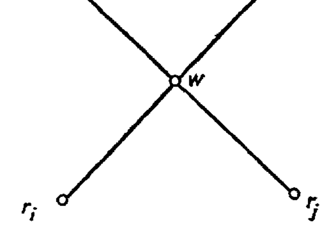

Fig. 5

Replacing r_i and r_j by r_i' and r_j' does not violate the cut condition. Moreover, any pair of edge-disjoint paths P_i', P_j' in the extended graph, where P_i' connects r_i' and s_i and P_j' connects r_j' and s_j, contains edge-disjoint paths P_i and P_j, where P_i connects r_i and s_i and P_j connects r_j and s_j.

Repeating this construction, we end up with $r_1, \ldots, r_t, s_1, \ldots, s_t$ occurring cyclically around I_1 (possibly after reordering indices and exchanging r_i and s_i). Similarly, we can assume that $r_{t+1}, \ldots, r_k, s_{t+1}, \ldots, s_k$ occur cyclically around I_2.

Now we can extend $\mathbf{R}^2 \setminus (I_1 \cup I_2)$ to the Klein bottle, by adding cross-caps along the boundaries of I_1 and I_2. We can extend G to a graph G' on the Klein bottle by adding edges $e_1, \ldots, e_k$ over the cross-caps, so that e_i connects r_i and s_i $(i=1, \ldots, \ldots, k)$. Then a circuit in G' is orientation-reversing if and only if it contains an odd number of edges from $e_1, \ldots, e_k$. The remainder of the proof is exactly the same as that of Theorem 4. ∎

Okamura's theorem has as special case the theorem of Okamura and Seymour [4], where $r_1, \ldots, r_k, s_1, \ldots, s_k$ are all on the boundary of one face.

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